

# A nonextensive entropy approach to kappa-distributions

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Most astrophysical plasmas are observed to have velocity distribution functions exhibiting non-Maxwellian suprathermal tails. The high energy particle populations are accurately represented by the family of kappa-distributions where the use of these distributions has been unjustly criticized because of a perceived lack of theoretical justification. We show that distributions very close to kappa-distributions are a consequence of the generalized entropy favored by nonextensive statistics, which provides the missing link for power-law models of non-thermal features from fundamental physics. With regard to the physical basis supplied by the Tsallis nonextensive entropy formalism we propose that this slightly modified functional form, qualitatively similar to the traditional kappa-distribution, be used in fitting particle spectra in the future.

# 1 Introduction

A variety of space observations indicate clearly the ubiquitous presence of suprathermal particle populations in astrophysical plasma environments [1]. The family of kappa velocity space distributions, introduced first by Vasyliunas [2], is recognized to be highly appropriate for modeling specific electron and ion components of different plasma states.

Numerous magnetospheric interaction processes and instabilities were studied successfully within the concept of kappa-distributions [3, 4] ranging from plasma sheet ion and electron spectra [5] to a combined electron-proton-hydrogen atom aurora [6]. The saturation of ring current particles towards a kappa-distribution was studied by Lui and Rostoker [7] and accurate fittings of observed electron flux spectra were performed by a power law at high energies [8]. Furthermore, it was demonstrated by Leubner [9] that the Jovian banded whistler mode emission can be interpreted within a kappa-distribution approach and that mirror instability thresholds are drastically reduced in suprathermal space plasmas [10, 11]. Energetic tail distributions play a key role in coronal plasma dynamics [12] and high resolution plasma observations near 1 AU confirm that even the distribution function of heavy solar wind ions are well fitted by a kappa-distribution [13].

The generation of velocity space distributions exhibiting pronounced energetic tails was frequently interpreted as a consequence of several different acceleration mechanisms. Those include besides DC parallel electric fields or field-aligned potential drops in reconnection regions also wave-particle interaction due to kinetic Alfvén wave turbulence [14, 15] and cyclotron interactions [16]. Clear evidence of the importance of suprathermal particle populations in space plasmas have motivated the development of a modified plasma dispersion function for kappa-distributions [17, 18].

The use of the family of kappa-distributions to model the observed non-thermal features of electron and ion structures was frequently criticized since a profound derivation in view of fundamental physics was not available. A classical analysis addressed to this problem was performed by Hasegawa [19] demonstrating that kappa-distributions turn out as consequence of the presence of suprathermal radiation fields in plasmas and Collier [20] considers the generation of kappa-like distributions using velocity space Lévy flights. Furthermore, a justification for the formation of power-law distributions in space plasmas due to electron acceleration by whistler mode waves was proposed by Ma and Summers [21, 22, 23] and a kinetic theory was developed showing that kappa-like velocity space distributions are a particular thermodynamic equilibrium state [24]. Here we demonstrate that the family of kappa-distributions results as consequence of the entropy generalization in nonextensive statistics, providing thus the missing link for the use of kappa-distributions from fundamental physics.

## 2 Theory

A generalization of the Boltzmann-Gibbs-Shannon entropy formula for statistical equilibrium was recognized to be required for systems subject to spatial or temporal long-range interactions making their behavior nonextensive. Any extensive formalism fails whenever a physical system includes long-range forces or long-range memory. In particular, this situation is usually found in astrophysical environments and plasma physics where, for example, the range of interactions is comparable to the size of the system considered. A generalized entropy is required to possess the usual properties of positivity, equiprobability, concavity and irreversibility but suitably extending the standard additivity to nonextensivity.

In view of the difficulties arising in this conjunction within the Boltzmann-Gibbs standard statistical mechanics motivated Tsallis [25, 26] to introduce a thermo-statistical theory based on the generalized entropy of the form

$$S_q = k_B \frac{1 - \sum p_i^q}{q - 1} \quad (1)$$

where  $p_i$  is the probability of the  $i^{th}$  microstate,  $k_B$  is Boltzmann's constant and  $q$  is a parameter quantifying the degree of nonextensivity and is commonly referred to as the entropic index. A crucial property of this entropy is the pseudoadditivity such that

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \quad (2)$$

for given subsystems  $A$  and  $B$  in the sense of factorizability of the microstate probabilities. For  $q \rightarrow 1$  the standard Boltzmann-Gibbs-Shannon extensive entropy is recovered as

$$S_q = -k_B \sum p_i \ln p_i \quad (3)$$

Applying the transformation

$$\frac{1}{q - 1} = -\kappa \quad (4)$$

to equation (1) and restricting to values  $-1 < q \leq 1$  yields the generalized entropy of the form

$$S_\kappa = \kappa k_B (\sum p_i^{1-1/\kappa} - 1) \quad (5)$$

for  $\frac{1}{2} < \kappa \leq \infty$ . With regard to Silva [27] one finds the corresponding one dimensional equilibrium velocity space distribution in kappa notation as

$$f(v) = A_\kappa \left[ 1 + \frac{1}{\kappa} \frac{v^2}{v_{th}^2} \right]^{-\kappa} \quad (6)$$

where the normalization constant reads

$$A_\kappa = \frac{N}{v_{th}} \frac{1}{\sqrt{\kappa}} \frac{\Gamma[\kappa]}{\Gamma[\kappa - 1/2]} \quad (7)$$

Here  $N$  denotes the particle density,  $v_{th} = \sqrt{2k_B T/m}$  is the thermal velocity where  $T$  and  $m$  are the temperature and the mass, respectively, of the species considered. The case  $\kappa = \infty$  corresponds to  $q = 1$  wherefrom the Maxwell equilibrium distribution is recovered. Figure 1a illuminates schematically the non-thermal behavior of the distribution function (6) for some values of the spectral index kappa.

Consistently, the one-dimensional equilibrium velocity space distribution in the q-nonextensive framework is written as

$$f(v) = A_q \left[ 1 - (q-1) \frac{v^2}{v_{th}^2} \right]^{1/(q-1)} \quad (8)$$

where

$$A_q = \frac{N}{v_{th}} \sqrt{1-q} \frac{\Gamma[1/(1-q)]}{\Gamma[1/(1-q) - 1/2]} \quad (9)$$

for  $-1 < q \leq 1$ . In analogy, the isotropic three dimensional velocity space distribution is found as

$$f(\mathbf{v}) = B_\kappa \left[ 1 + \frac{1}{\kappa} \frac{v^2}{v_{th}^2} \right]^{-\kappa} \quad (10)$$

identical to the distribution function for a plasma in a suprathermal radiation field discussed by Hasegawa (1985) where the normalization constant reads

$$B_\kappa = \frac{N}{\pi^{3/2} v_{th}^3} \frac{1}{\kappa^{3/2}} \frac{\Gamma[\kappa]}{\Gamma[\kappa - 3/2]} \quad (11)$$

for  $\frac{3}{2} < \kappa \leq \infty$ . Equation (6) and (10) denote a reduced form of the standard kappa-distribution used for astrophysical applications and written conventionally as

$$f(\mathbf{v}) = A_\kappa \left[ 1 + \frac{1}{\kappa} \frac{v^2}{v_{th}^2} \right]^{-(\kappa+1)} \quad (12)$$

with the normalization constant

$$A_\kappa = \frac{N}{\pi^{3/2} v_{th}^3} \frac{1}{\kappa^{3/2}} \frac{\Gamma[\kappa + 1]}{\Gamma[\kappa - 1/2]} \quad (13)$$

For a variation of the spectral index kappa Figure 1 demonstrates the different behavior between the one dimensional case of the distribution function (12), generally used for

space applications and modeling, as compared to equation (6) resulting from the modified entropy approach. Since (6) and (12) differ only in the exponent no exact mapping is possible when substituting  $\kappa \rightarrow \kappa + 1$ . Nevertheless, highly similar structures are found for both with the appropriate values of kappa, hence favoring relation (6) for astrophysical velocity space distribution modeling due to the physical background.

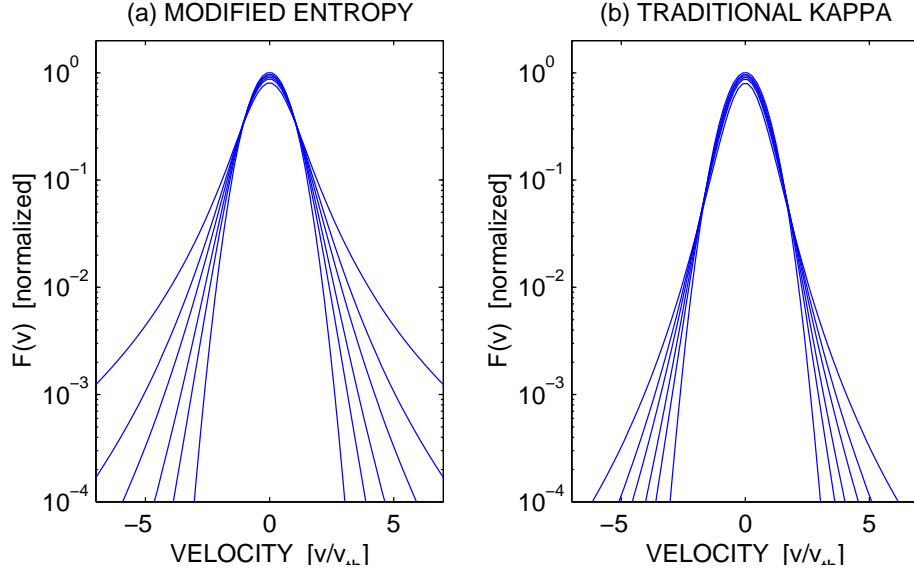


Figure 1: Schematic plot of the family of kappa-distributions. The subplots (a) and (b) demonstrate the one-dimensional functional dependence of equation (6) and (12), respectively. For both,  $\kappa = 2$  corresponds to the outermost curve followed by values  $\kappa = 3, 4, 6$  and  $10$ . The innermost curve represents with  $\kappa = \infty$  an isotropic Maxwellian.

An effective thermal speed  $\theta = v_{th} \sqrt{(\kappa - 3/2)/\kappa}$  is commonly defined from the moments of the distribution function (12) where the spectral index kappa is conventionally limited to positive values  $\kappa > 3/2$  and we note that negative values of kappa have never been considered for space plasma applications. Contrary, the generalization to negative values of kappa identical to  $q \geq 1$ , discussed in the framework of nonextensive statistics as well, generates according to equation (6) or (10) a thermal cutoff at the maximum allowed velocity

$$v_{\max} = \sqrt{\kappa \frac{2k_B T}{m}} \quad (14)$$

hence providing an additional physical interpretation of the spectral index kappa of non-thermal plasma structures. To our knowledge there is presently no observational evidence for distributions corresponding to  $q \geq 1$  in space plasmas available. On the other hand it should be mentioned that any observed thermal cutoff in velocity distribu-

tions would not have been recognized or interpreted as being a consequence of a possible nonextensivity of the system.

Finally we note that Collier [28] has evaluated the Boltzmann entropy for kappa-distribution functions. Upon introducing the corresponding nonextensive velocity space distribution the resulting kappa-dependent entropy is found to obey qualitatively the same functional dependence. Since the suprathermal tails are more pronounced in the modified entropy case as compared to the standard kappa function, see Figure 1, also the entropy enhancement for low values of the spectral index, corresponding to a large fraction of suprathermal particle populations, is increased relatively to the traditional kappa-distribution case. Both approaches merge for large values of kappa towards the Maxwellian limit.

### 3 Conclusions

Nonextensive statistics was successfully applied to a number of astrophysical and cosmological scenarios. Those include stellar polytropes [29], the solar neutrino problem [30], peculiar velocity distributions of galaxies [31] and generally systems with long range interactions and fractal like space-times. Cosmological implications were discussed [32] and recently an analysis of plasma oscillations in a collisionless thermal plasma was provided from q-statistics [33]. On the other hand, kappa-distributions are highly favored in any kind of space plasma modeling [1] among others, where a reasonable physical background was not apparent. A comprehensive discussion of kappa distributions in view of experimentally favored non-thermal tail formations is provided by Leubner and Schupfer [10] where also typical values of the index  $\kappa$  are quoted and referenced for different space plasma environments.

In the present analysis the missing link to fundamental physics is provided within the framework of an entropy modification consistent with nonextensive statistics. The family of kappa distributions are obtained from the positive definite part  $\frac{1}{2} \leq \kappa \leq \infty$ , corresponding to  $-1 \leq q \leq 1$  of the general statistical formalism where in analogy the spectral index kappa is a measure of the degree of nonextensivity. Since the main theorems of the standard Maxwell-Boltzmann statistics admit profound generalizations within nonextensive statistics [34, 35, 36, 37, 38], a justification for the use of kappa-distributions in astrophysical plasma modeling is provided from fundamental physics.

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